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Application of the Faddeev-Jackiw formalism to the gauged WZW model

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Abstract

The two-flavour Wess-Zumino model coupled to electromagnetism is treated as a constraint system using the Faddeev-Jackiw method. Expanding into series of powers of the pion fields and keeping terms up to second and third order we obtain Coulomb-gauge Lagrangeans containing non-local terms.

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1 Introduction

The Faddeev-Jackiw method [1] avoids the separation of the constraints into first and second class and gives us a simple and straightforward way to deal with constraint systems. A brief outline of this technique is given below. Let $H(p,q)$ be the Hamiltonian describing the dynamics of a certain system. We can always construct a first order in time derivatives Lagrangean whose configuration space coincides with the Hamiltonian phase space. This can be done by enlarging the n dimensional configuration space to a $2n$ dimensional configuration space. We define the new coordinates ξ^i , as follows

$$\begin{aligned} \xi^i &= p_i \,, \qquad && i = 1, ..., n \\ \xi^i &= q^i \,, \qquad && i = n+1, ..., 2n \end{aligned}$$

Then the Lagrangean of the system can be written as

$$\mathcal{L}(\xi, \dot{\xi}) = \frac{1}{2}\xi^i \omega_{ij} \dot{\xi}^j - H(\xi) \;\; , \tag{1}$$

after dropping a total time derivative. The matrix ω_{ij} is given by

$$\omega_{ij} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}_{ij}$$

and has an inverse ω^{ij}. The system of the example has no constraint and the evolution of ξ is given as follows using the Euler-Lagrange equations

$$\dot{\xi}^i = \omega^{ij} \frac{\partial}{\partial \xi^j} H(\xi) \;\; . \tag{2}$$

For a general unconstrained system described by the Lagrangean

$$\mathcal{L}(\xi, \dot{\xi}) = a_i(\xi) \dot{\xi}^i - H(\xi) \,, \qquad i = 1, ..., N \tag{3}$$

with arbitrary $a_i(\xi)$, the Euler-Lagrange equations are given by

$$f_{ij} \dot{\xi}^j = \frac{\partial}{\partial \xi_i} H \;\; , \tag{4}$$

where $f_{ij} = \frac{\partial}{\partial \xi_i} a_j - \frac{\partial}{\partial \xi_j} a_i$ is an invertible $2n \times 2n$ matrix. Then according to Darboux's theorem we can construct a coordinate transformation $Q^i(\xi)$ so

that the canonical one-form $a_i(\xi)d\xi^i$ defined in the first part of (3) acquires the "diagonal" form given in (1)

$$a_i(\xi)d\xi^i = \frac{1}{2}Q^k(\xi)\omega_{kl}dQ^l(\xi) \ . \tag{5}$$

In the case that the Lagrangean (3) describes a constraint system the matrix f_{ij} is singular. The Darboux's theorem can still be applied for the maximal $2n \times 2n$ nonsingular subblock of f_{ij}. The Lagrangean (3) transforms as follows

$$\mathcal{L}(Q,\dot{Q},z) = \frac{1}{2}Q^k\omega_{kl}\dot{Q}^l - H'(Q,z) \ , \qquad k,l = 1,...,2n \tag{6}$$

where z denote the $N-2n$ coordinates that are left unchanged. Some of the z_i may appear non-linearly and some linearly in $H'(Q,z)$. Then using the Euler-Lagrange equation for these coordinates we can solve for as many z_i's as possible in term of Q^i's and other z_i's and replace back in $H'(Q,z)$ so finally we are left only with linearly occuring z_i's. After this step is completed the Lagrangean (6) can be written as

$$\mathcal{L} = \frac{1}{2}Q^k\omega_{kl}\dot{Q}^l - V(Q) - z_i\Phi^i(Q) \ \ , \tag{7}$$

where we see that z_i become the Lagrange multipliers and $\Phi^i(Q)$ are the constraints. By solving the equations of constraints and replacing back in (7) we reduce the number of Q's and we end up with a Lagrangean which has the structure given in (3). Then the whole procedure can be repeated again until all constraints are eliminated and we are left with an unconstrained Lagrangean whose canonical one form is diagonal. Now the canonical quantization rules can be applied to the new canonical coordinates, which we rename $p_i \ , \ q^i \ , \ i = 1,...,n$.

The FJ approach to constraint systems has by now been successfully used in various fields [5, 6, 7, 8, 9, 10, 11, 12, 13] . Our intention is to apply it to the two-flavour WZW model coupled to electromagnetism. This model describes very well the low energy interactions among pions and pion with photons including those related to the axial anomaly.

2 Expanding up to second order in the pion fields

The effective action of the model [2, 3, 4] given by

$$\begin{aligned}
\Gamma_{eff}(U,A_\mu) &= \Gamma_{\rm EM}(A_\mu)+\Gamma_\sigma(U,A_\mu)+\Gamma_{\rm WZW}(U,A_\mu) \ , \\
\Gamma_{\rm EM}(A_\mu) &= -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \ , \\
\Gamma_\sigma(U,A_\mu) &= -\frac{f_\pi^2}{16}\int d^4x {\rm tr}\,(R_\mu R^\mu) \\
&= -\frac{f_\pi^2}{16}\int d^4x {\rm tr}\,(r_\mu r^\mu)+\frac{if_\pi^2 e}{8}\int d^4x A_\mu {\rm tr}\,[Q(r_\mu - l_\mu)] \\
&+\frac{f_\pi^2 e^2}{8}\int d^4x A_\mu A^\mu {\rm tr}\,(Q^2 - U^\dagger Q U Q) \ , \\
\Gamma_{\rm WZW}(U,A_\mu) &= -\frac{N_c e}{48\pi^2}\int d^4x \epsilon^{\mu\nu\alpha\beta}A_\mu {\rm tr}\,[Q(r_\nu r_\alpha r_\beta + l_\nu l_\alpha l_\beta)] \\
&+\frac{iN_c e^2}{24\pi^2}\int d^4x \epsilon^{\mu\nu\alpha\beta}A_\mu(\partial_\nu A_\alpha){\rm tr}\,[Q^2(r_\beta + l_\beta) \\
&+\frac{1}{2}QU^\dagger QU r_\beta + \frac{1}{2}QUQU^\dagger l_\beta] \ ,
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
&U = \exp\,(2i\theta_a\tau_a/f_\pi) \ , \quad r_\mu = U^\dagger\partial_\mu U \ , \quad R_\mu = U^\dagger D_\mu U \ , \\
&l_\mu = (\partial_\mu U)U^\dagger \ , \quad L_\mu = (D_\mu U)U^\dagger \ .
\end{aligned}$$

See Appendix for notation. In the three flavor WZW model the term

$$\Gamma_{WZW}(U) = -\frac{iN_c}{240\pi^2}\int d^5x \epsilon^{ijklm}{\rm tr}\,(l_i l_j l_k l_l l_m) \ ,$$

should also be included but this term vanishes in our case. The effective action (8) is by construction gauge invariant under $U_{EM}(1)$ gauge transformation. Applying the FJ method to the full case seems quite difficult so we use expansion into powers of the pions fields θ_a

$$U = 1 + \frac{2i}{f_\pi}\theta_a\tau_a + \ldots \tag{9}$$

We substitute into (8) and keep terms up to 2^{nd} order. After calculating the traces the effective Lagrangean of the two-flavor Wess-Zumino model coupled to electromagnetism is written as follows

$$
\begin{aligned}
\mathcal{L}_{eff} &= \mathcal{L}_{EM} + \mathcal{L}_{\sigma}^{(2)} + \mathcal{L}_{WZW}^{(2)} + O(\theta^3) \quad , \qquad (10)\\
\mathcal{L}_{EM} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \\
\mathcal{L}_{\sigma}^{(2)} &= \frac{1}{2}\partial_\mu\theta_a\partial^\mu\theta_a + eA^\mu(\theta_2\partial_\mu\theta_1 - \theta_1\partial_\mu\theta_2) + \frac{e^2}{2}A_\mu A^\mu(\theta_1^2+\theta_2^2) \quad , \\
\mathcal{L}_{\mathrm{WZW}}^{(2)} &= -\frac{N_c e^2}{12\pi^2 f_\pi}\epsilon^{\mu\nu\alpha\beta}A_\mu(\partial_\nu A_\alpha)\partial_\beta\theta_3 \quad .
\end{aligned}
$$

Then in the non-covariant notation

$$
\begin{aligned}
\mathcal{L}_{eff} = &-\mathbf{E}\cdot\dot{\mathbf{A}} + \frac{1}{2}\dot{\theta_a}^{\,2} + \mathrm{eA}_0(\theta_2\dot{\theta}_1 - \theta_1\dot{\theta}_2) + \mathrm{A}_0\nabla\cdot\mathbf{E} \\
&+ \frac{e^2}{2}(A_0^2 - \mathbf{A}^2)(\theta_1^2+\theta_2^2) - \frac{1}{2}(\mathbf{E^2} + \mathbf{B^2} + (\nabla\theta_a)^2) \\
&+ e\mathbf{A}\cdot(\theta_2\nabla\theta_1 - \theta_1\nabla\theta_2) + \frac{\mathrm{N_c e^2}}{6\pi^2\mathrm{f}_\pi}(\mathbf{E}\cdot\mathbf{B})\theta_3 \quad , \qquad (11)
\end{aligned}
$$

where $\mathbf{E} = -\dot{\mathbf{A}} - \nabla\mathrm{A}_0, \quad \mathbf{B} = \nabla\times\mathbf{A}$.

The canonical momenta conjugate to θ_a are given by

$$
\begin{aligned}
p_1 &= \tfrac{\partial\mathcal{L}_{eff}}{\partial\dot{\theta}_1} = \dot{\theta}_1 + eA_0\theta_2 \quad , \\
p_2 &= \tfrac{\partial\mathcal{L}_{eff}}{\partial\dot{\theta}_2} = \dot{\theta}_2 - eA_0\theta_1 \quad , \\
p_3 &= \tfrac{\partial\mathcal{L}_{eff}}{\partial\dot{\theta}_3} = \dot{\theta}_3 \quad .
\end{aligned}
$$

In the enlarged configuration space with coordinates $-E_a$, A_a , p_a , θ_a , $a = 1,2,3$ the effective Lagrangean (11) can be written after dropping a total time derivative as an expression first order in time derivatives of the coordinates, having the structure given in (7)

$$
\begin{aligned}
\mathcal{L}_{eff} &= -\mathbf{E}\cdot\dot{\mathbf{A}} + \mathrm{p}_a\dot{\theta_a} - \mathrm{H}^{(2)}(\mathbf{E},\mathbf{A},\mathrm{p}_a,\theta_a) - \mathrm{A}_0(\rho^{(2)} - \nabla\cdot\mathbf{E}) + \mathrm{O}(\theta^3) \quad , (12)\\
H^{(2)} &= \frac{1}{2}[\mathbf{E^2} + \mathbf{B^2} + (\nabla\theta_a)^2 + \mathrm{p}_a^2] + \mathrm{e}\mathbf{A}\cdot(\theta_1\nabla\theta_2 - \theta_2\nabla\theta_1) + \frac{\mathrm{e}^2}{2}\mathbf{A^2}(\theta_1^2+\theta_2^2) \\
&\quad - \frac{N_c e^2}{6\pi^2 f_\pi}(\mathbf{E}\cdot\mathbf{B})\theta_3 \quad , \\
\rho^{(2)} &= e(p_2\theta_1 - p_1\theta_2) \quad .
\end{aligned}
$$

We see that the scalar potential A_0 is the Lagrange multiplier and $\rho^{(2)} - \nabla \cdot \mathbf{E}$ is the constraint. In order to solve the equation of the constraint

$$\nabla \cdot \mathbf{E} - \rho^{(2)} = 0 \quad , \tag{13}$$

we decompose the electric field $\mathbf{E}$ and the vector potential $\mathbf{A}$ into transverse and longitudinal components

$$\mathbf{E} = \mathbf{E}_{\rm T} + \mathbf{E}_{\rm L} \ , \qquad \mathbf{A} = \mathbf{A}_{\rm T} + \mathbf{A}_{\rm L} \ ,$$

$$\nabla \cdot \mathbf{E}_{\rm T} = 0 \ , \ \nabla \times \mathbf{E}_{\rm L} = 0 \ , \ \nabla \cdot \mathbf{A}_{\rm T} = 0 \ , \ \nabla \times \mathbf{A}_{\rm L} = 0 \ \ .$$

Then (13) implies that

$$\mathbf{E}_{\rm L} = \frac{\nabla}{\nabla^2}\rho^{(2)} \quad . \tag{14}$$

Substituting into (12) we obtain apart from total spatial derivatives

$$\begin{aligned} \mathcal{L}_{eff} = & -\mathbf{E}_{\rm T} \cdot \dot{\mathbf{A}}_{\rm T} + \rho^{(2)}\frac{\nabla}{\nabla^2} \cdot \dot{\mathbf{A}}_{\rm L} + {\rm p}_a\dot{\theta}_a \\ & - \frac{1}{2}[\mathbf{E}_{\rm T}^{\mathbf{2}} + \mathbf{B}^{\mathbf{2}} - \rho^{(2)}\frac{1}{\nabla^2}\rho^{(2)} + (\nabla\theta_a)^2 + {\rm p}_a^2] \\ & - e\mathbf{A} \cdot (\theta_1\nabla\theta_2 - \theta_2\nabla\theta_1) - \frac{{\rm e}^2}{2}\mathbf{A}^2(\theta_1^2 + \theta_2^2) + \frac{{\rm N_c e^2}}{6\pi^2{\rm f}_\pi}(\mathbf{E} \cdot \mathbf{B})\theta_3 \ \ . \end{aligned} \tag{15}$$

We see that the longitudinal part of the vector potential $\mathbf{A}_{\rm L}$, enters the canonical one-form of the Lagrangean in an uncanonical way. In order to diagonalize the canonical one-form in (15) we perform the following Darboux's transformations

$$\begin{aligned} & p_1 \to p_1 \cos\alpha + p_2 \sin\alpha \ , \qquad \theta_1 \to \theta_1 \cos\alpha + \theta_2 \sin\alpha \ , \\ & p_2 \to p_2 \cos\alpha - p_1 \sin\alpha \ , \qquad \theta_2 \to \theta_2 \cos\alpha - \theta_1 \sin\alpha \ , \\ & p_3 \to p_3 \ , \ \theta_3 \to \theta_3 \ , \ \mathbf{E}_{\rm T} \to \mathbf{E}_{\rm T} \ , \ \mathbf{A}_{\rm T} \to \mathbf{A}_{\rm T} \ , \end{aligned} \tag{16}$$

where $\alpha = e\frac{\nabla}{\nabla^2} \cdot \mathbf{A}_{\rm L}$ This choice of coordinate transformations leads to the cancelation of $\mathbf{A}_{\rm L}$ and fixes the gauge of the Lagrangean to the Coulomb-gauge

$$\mathcal{L}_{eff} = -\mathbf{E}_{\rm T} \cdot \dot{\mathbf{A}}_{\rm T} + {\rm p}_a\dot{\theta}_a - {\rm H}_{\rm C}^{(2)}(\mathbf{E}_{\rm T}, \mathbf{A}_{\rm T}, {\rm p}_a, \theta_a) \ \ , \tag{17}$$

$$
\begin{aligned}
H_C^{(2)}(\mathbf{E}_{\mathrm{T}}, \mathbf{A}_{\mathrm{T}}, \mathrm{p}_a, \theta_a) = & \frac{1}{2}[\mathbf{E}_{\mathrm{T}}^{\mathbf{2}} + \mathbf{B}^{\mathbf{2}} - \rho^{(2)} \frac{1}{\nabla^2} \rho^{(2)} + (\nabla \theta_a)^2 + \mathrm{p}_a^2] \\
& + e\mathbf{A}_{\mathrm{T}} \cdot (\theta_1 \nabla \theta_2 - \theta_2 \nabla \theta_1) + \frac{\mathrm{e}^2}{2} \mathbf{A}_{\mathrm{T}}^{\mathbf{2}} (\theta_1^2 + \theta_2^2) \\
& - \frac{N_c e^2}{6\pi^2 f_\pi} [\mathbf{E}_{\mathrm{T}} \cdot \mathbf{B} + (\frac{\nabla}{\nabla^2} \rho^{(2)}) \cdot \mathbf{B}] \theta_3 \quad .
\end{aligned}
$$

We see that only the physical transverse components of the vector potential enter in the expression of the Lagrangean. (Note that $\mathbf{B} = \nabla \times \mathbf{A}_{\mathrm{T}}$).

3 Keeping third order terms

We now proceed with the expansion and keep terms up to third order in the Goldstone boson fields

$$
\mathcal{L}_{eff} = \mathcal{L}_{EM} + \mathcal{L}_{\sigma}^{(2)} + \mathcal{L}_{WZW}^{(2)} + \mathcal{L}_{WZW}^{(3)} + O(\theta^4) \quad , \tag{18}
$$

where the expressions for $\mathcal{L}_{\sigma}^{(2)}$ and $\mathcal{L}_{WZW}^{(2)}$ are given in (10) and

$$
\begin{aligned}
\mathcal{L}_{WZW}^{(3)} = & -\frac{N_c e}{3\pi^2 f_\pi^3} \epsilon^{\mu\nu\alpha\beta} (\partial_\mu A_\nu)(\theta_1 \partial_\alpha \theta_2 - \theta_2 \partial_\alpha \theta_1) \partial_\beta \theta_3 \\
& + \frac{2N_c e^2}{9\pi^2 f_\pi^3} \epsilon^{\mu\nu\alpha\beta} (\partial_\mu A_\nu)(\partial_\alpha A_\beta)(\theta_1^2 + \theta_2^2) \theta_3 \\
& - \frac{N_c e^2}{3\pi^2 f_\pi^3} \epsilon^{\mu\nu\alpha\beta} A_\mu (\partial_\nu A_\alpha) \theta_3 \partial_\beta (\theta_1^2 + \theta_2^2) \quad .
\end{aligned}
$$

We see that the Wess-Zumino-Witten term is the only term that contributes to this order. Next we derive the expressions for the canonical momenta p_a conjugate to the pion fields θ_a from the new Lagrangean (18). Then after long mathematical manipulations and dropping total derivatives we obtain the following expression for $\mathcal{L}_{eff}$ in the enlarged configuration space

$$
\mathcal{L}_{eff} = -\mathbf{E} \cdot \dot{\mathbf{A}} + \mathrm{p}_a \dot{\theta}_a - \mathrm{H}^{(2)} - \mathrm{H}^{(3)} - \mathrm{A}_0(\rho^{(2)} + \rho^{(3)} - \nabla \cdot \mathbf{E}) + \mathrm{O}(\theta^4) \quad , \tag{19}
$$

where the expression for $H^{(2)}$ is given in (12) and

$$
\begin{aligned}
H^{(3)} =& -\frac{N_c e}{3\pi^2 f_\pi^3}(\mathbf{E}\times\nabla\theta_3 - \mathrm{p}_3\mathbf{B})\cdot(\theta_1\nabla\theta_2 - \theta_2\nabla\theta_1) \\
&+\frac{4N_c e^2}{9\pi^2 f_\pi^3}(\mathbf{E}\cdot\mathbf{B})(\theta_1^2+\theta_2^2)\theta_3 - \frac{\mathrm{N_c e^2}}{3\pi^2 \mathrm{f}_\pi^3}(\mathbf{E}\times\mathbf{A})\cdot[\nabla(\theta_1^2+\theta_2^2)]\theta_3 \\
&-\frac{2N_c e^2}{3\pi^2 f_\pi^3}(\mathbf{A}\cdot\mathbf{B})(\mathrm{p}_1\theta_1+\mathrm{p}_2\theta_2)\theta_3 - \frac{\mathrm{N_c e}}{3\pi^2 \mathrm{f}_\pi^3}(\mathbf{B}\cdot\nabla\theta_3)(\mathrm{p}_2\theta_1 - \mathrm{p}_1\theta_2) \ , \\
\rho^{(2)} =& \, e(p_2\theta_1 - p_1\theta_2) \ , \\
\rho^{(3)} =& -\frac{N_c e^2}{3\pi^2 f_\pi^3}\nabla\cdot[\mathbf{B}(\theta_1^2+\theta_2^2)\theta_3] \ ,
\end{aligned}
$$

A_0 is again the Lagrange multiplier and the equation of the constraint is given by

$$\nabla\cdot\mathbf{E} - (\rho^{(2)}+\rho^{(3)}) = 0 \ . \tag{20}$$

This equation has similar structure as in the previous case (13) with only an extra term of the third order in the pion fields in the divergence. So we proceed similarly and decompose $\mathbf{E}$ and $\mathbf{A}$ into transverse and longitudinal components.Then (20) implies that

$$\mathbf{E}_\mathrm{L} = \frac{\nabla}{\nabla^2}(\rho^{(2)}+\rho^{(3)}) \ .$$

Substituting into (19) we end up with a Lagrangean first order in time derivatives of the coordinates whose canonical one-form

$$-\mathbf{E}_\mathrm{T}\cdot\dot{\mathbf{A}}_\mathrm{T} + \rho^{(2)}\frac{\nabla}{\nabla^2}\cdot\dot{\mathbf{A}}_\mathrm{L} + \rho^{(3)}\frac{\nabla}{\nabla^2}\cdot\dot{\mathbf{A}}_\mathrm{L} + \mathrm{p}_a\dot{\theta}_a \ ,$$

must be diagonalized. To do this we proceed in two steps. First we perform the Darboux's transformations given in (16) as in the previous case. This transformations lead to partial diagonalization. We obtain a new Lagrangean whose canonical one-form is given by

$$-\mathbf{E}_\mathrm{T}\cdot\dot{\mathbf{A}}_\mathrm{T} + \rho^{(3)}\frac{\nabla}{\nabla^2}\cdot\dot{\mathbf{A}}_\mathrm{L} + \mathrm{p}_a\dot{\theta}_a \ ,$$

In order to proceed with the second step we must write the term $\rho^{(3)}\frac{\nabla}{\nabla^2}\cdot\dot{\mathbf{A}}_\mathrm{L}$ in a more suitable way

$$\rho^{(3)}\frac{\nabla}{\nabla^2}\cdot\dot{\mathbf{A}}_\mathrm{L} = \mathrm{tot.der.} - \frac{\mathrm{N_c e^2}}{3\pi^2\mathrm{f}_\pi^3}[(\nabla\phi\times\mathbf{A}_\mathrm{L})\cdot\dot{\mathbf{A}}_\mathrm{T} + (\mathbf{B}\cdot\mathbf{A}_\mathrm{L})\dot{\phi}] \ ,$$

where $\phi = (\theta_1^2 + \theta_2^2)\theta_3$.

Now we perform the second transformation

$$
\begin{aligned}
&\mathbf{E}_{\mathrm{T}} \to \mathbf{E}_{\mathrm{T}} - \tfrac{\mathrm{N_c e^2}}{3\pi^2 \mathrm{f}_\pi^3}\nabla[(\theta_1^2+\theta_2^2)\theta_3] \times \mathbf{A}_{\mathrm{L}} \ , \\
&p_1 \to p_1 + \tfrac{2N_c e^2}{3\pi^2 f_\pi^3}(\mathbf{B}\cdot\mathbf{A}_{\mathrm{L}})\theta_1\theta_3 \ , \\
&p_2 \to p_2 + \tfrac{2N_c e^2}{3\pi^2 f_\pi^3}(\mathbf{B}\cdot\mathbf{A}_{\mathrm{L}})\theta_2\theta_3 \ , \\
&p_3 \to p_3 + \tfrac{N_c e^2}{3\pi^2 f_\pi^3}(\mathbf{B}\cdot\mathbf{A}_{\mathrm{L}})(\theta_1^2+\theta_2^2) \ , \\
&\mathbf{A}_{\mathrm{T}} \to \mathbf{A}_{\mathrm{T}} \ , \ \theta_1 \to \theta_1 \ , \ \theta_2 \to \theta_2 \ , \ \theta_3 \to \theta_3 \ .
\end{aligned}
$$

This transformation completes the diagonalization of the canonical one-form and we end up with a Lagrangean where the longitudinal part of the vector potential A_{L} cancels out exactly

$$
\mathcal{L} = -\mathbf{E}_{\mathrm{T}}\cdot\dot{\mathbf{A}}_{\mathrm{T}} + \mathrm{p}_a\dot{\theta}_a - \mathrm{H}_{\mathrm{C}}^{(2)} - \mathrm{H}_{\mathrm{C}}^{(3)} + \mathrm{O}(\theta^4) \ , \qquad (21)
$$

where

$$
\begin{aligned}
H_C^{(3)} = &-\frac{N_c e}{3\pi^2 f_\pi^3}(\mathbf{E}_{\mathrm{T}}\times\nabla\theta_3 - \mathrm{p}_3\mathbf{B})\cdot(\theta_1\nabla\theta_2 - \theta_2\nabla\theta_1) \\
&+\frac{4N_c e^2}{9\pi^2 f_\pi^3}(\mathbf{E}_{\mathrm{T}}\cdot\mathbf{B})(\theta_1^2+\theta_2^2)\theta_3 - \frac{\mathrm{N_c e^2}}{3\pi^2\mathrm{f}_\pi^3}(\mathbf{E}_{\mathrm{T}}\times\mathbf{A}_{\mathrm{T}})\cdot[\nabla(\theta_1^2+\theta_2^2)]\theta_3 \\
&-\frac{2N_c e^2}{3\pi^2 f_\pi^3}(\mathbf{A}_{\mathrm{T}}\cdot\mathbf{B})(\mathrm{p}_1\theta_1+\mathrm{p}_2\theta_2)\theta_3 - \frac{\mathrm{N_c e}}{3\pi^2\mathrm{f}_\pi^3}(\mathbf{B}\cdot\nabla\theta_3)(\mathrm{p}_2\theta_1 - \mathrm{p}_1\theta_2) \ ,
\end{aligned}
$$

and the expression for $H_C^{(2)}$ is the given in (17).

We see that application of the Faddeev-Jackiw method for constraint systems to the two flavour Wess-Zumino-Witten model coupled to electromagnetism leads up to second and third order in the pion fields to Coulomb-gauge Lagrangeans including nonlocal interaction terms as given in (17). Futher application of the method to the three flavour case is currently under investigation.

4 Appendix

Our metric is $g_{\mu\nu} = diag(1,-1,-1,-1)$, $Q = diag(2/3,-1/3)$ is the charge matrix, $D_\mu = \partial_\mu + ieA_\mu[Q, \ ]$ denote the covariant derivative. By τ_a , $a =$

$1, 2, 3$ we denote Pauli matrices. We choose $e > 0$ so that the electric charge of the electron is $-e$. We define $\epsilon^{0123} = 1$.

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